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Alternative solution for cross-ply laminated composite thick plates

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Alternative solution for cross-ply laminated composite thick plates

The structural analysis of orthotropic and linear elastic cross-ply laminated composite thick plates is made by considering complete effects of transverse shear and normal stresses. The solutions are obtained using the functional analysis method in conjunction with the Gâteaux differential. A new functional with boundary condition terms is developed for the analysis. A four-node serendipity element with eight degrees of freedom is used. To ensure the accuracy of the developed mixed finite element (MFE), numerical results are compared with those of the published solutions. It has been proven that the developed MFE is highly accurate and efficient.

Key words:
cross-ply laminate, composite thick plates, mixed finite element, static analysis, Gâteaux differential

Izvorni znanstveni rad

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Alternativno rješenje za križno uslojene kompozitne debele ploče

U radu se opisuje analiza ortotropnih linearnoelastičnih križno uslojenih kompozitnih debelih ploča, pri čemu se u obzir uzima zajedničko djelovanje poprečnih posmičnih i normalnih naprezanja. Rješenja se dobivaju primjenom metoda funkcionalne analize u kombinaciji s Gâteauxovom derivacijom. Za potrebe analize razvijen je novi funkcional sa članovima za rubne uvjete. Primijenjen je serendipity element sa četiri čvora i osam stupnjeva slobode. Kako bi se potvrdila točnost novorazvijenoga mješovitog konačnog elementa , numerički su rezultati uspoređeni s objavljenim rješenjima. Utvrđeno je da je novorazvijeni element izuzetno točan i djelotvoran.

Ključne riječi:
križno uslojeni laminat, kompozitne ploče debljeg presjeka, mješoviti konačni element, statička analiza, Gâteauxova derivacija

Wissenschaftlicher Originalbeitrag

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Alternativlösung für starke quer geschichtete Verbundplatten

In dieser Arbeit wird die Analyse orthotropischer linear elastischer quer geschichteter starker Verbundplatten beschrieben. Dabei wird die Wechselwirkung von Schub- und Normalspannungen berücksichtigt. Die Lösungen werden aufgrund von funktionalanalytischen Methoden in der Kombination mit der Gâteaux-Ableitung aufgezeigt. Für die Analyse wurde ein neues Funktional mit Gliedern zur Beschreibung der Randbedingungen entwickelt. Ein Serendipity-Element mit vier Knoten und acht Freiheitsgraden wurde implementiert. Um die Genauigkeit des neuen gemischten Finiten Elementes zu prüfen, wurden die numerischen Resultate mit veröffentlichten Resultaten verglichen. Eine hohe Genauigkeit und Effizienz des neu entwickelten Elements konnte festgestellt werden.

Schlüsselwörter:
quer geschichtetes Laminat, starke Verbundplatten, gemischte Finite Elemente, statische Analyse, Gâteaux-Ableitung

1. Introduction

Structural components made with composite materials are extensively being used in various engineering applications due to their high strength/weight and stiffness/weight ratios. Mechanical behaviour of composite materials has been of interest to many researchers [1-2].

A considerable focus has recently been placed on understanding the response of laminated structural components. Many calculation models for the analysis of laminated composite plates are available in the literature. The authors of the paper [3] analysed orthotropic plates based on three theories (Kirchhoff, Reissner-Mindlin and Higher-order theory) of the segmentation method for the solution of orthotropic rectangular plates simply supported on two opposite edges and with any other boundary conditions on the other two ends. Static behaviour of orthotropic rectangular thick plates is examined in [4] using initial functional methods to analyse uniformly loaded simply supported square plates for various thickness and material properties. The mixed finite element method is used in [5] for the bending problem of clamped anisotropic / orthotropic / isotropic plates with variable/ constant thickness. A mesh-free method is used in [6] for the static and free vibration analysis of shear deformable laminated composite plates with various side to thickness ratios, material coefficients, boundary conditions or ply angles. Field variables were represented by a set of properly scattered nodes, and the element connectivity was satisfied without any requirement. A penalty technique, which performs with higher efficiency than Lagrange multipliers and the orthogonal transformation method, was used to impose essential boundary conditions. A global–local theory with the three-dimensional elasticity correction for stress and deformation analysis of multilayered and sandwich plates is presented in [7]. Based on a Galerkin-type finite element procedure, the governing equations were solved employing second-order Lagrangian elements. A simple mixed quadrilateral 3D plate element is proposed in [8]. It is particularly suitable for stress resultant recovery in the case of coarse meshes, with six degrees of freedom at each node for the linear static and buckling analysis of plate. A new curvature tensor is developed in [9] for the composite laminated Reddy plate based on the new modified couple stress theory. A simply supported cross-ply laminated plate subjected to loads of sinus function was solved by applying the developed plate model. A simple C° isoperimetric finite element formulation is presented in [10]. It is based on the refined higher-order shear deformation theory for the bending and free vibration analysis of composite and sandwich laminated plates. The proposed theory accounts for parabolic distribution of the transverse shear strains through thickness of the plate and rotary inertia effects. The effects of the degree of orthotropy, length to thickness ratio and plate aspect ratio upon the fundamental frequencies, are discussed. The static free vibration and buckling analysis of laminated composite plates based on a four-variable refined plate theory is studied in [11]. The presented method gives good results by just using four degrees of freedom in each node.

A considerable research has been conducted for the analysis of laminated composite plates based on various shear deformation theories. A simple first order shear deformation theory for the bending and free vibration analysis of simply supported antisymmetric cross-ply and angle-ply laminated composite plates is proposed in [12]. Unlike the existing first-order shear deformation theory, the presented one contains only four unknowns and has strong similarities with the classical plate theory. Equations of motion and boundary conditions were derived from Hamilton's principle. Analytical solutions were obtained for simply supported antisymmetric cross-ply and angle-ply laminates. The formulation of an enriched plate element according to the first order shear deformation theory is presented in [13]. To take into account plates of several geometrical shapes, an arbitrary quadrilateral laminate was mapped onto a basic square laminate. The static and buckling distribution of transverse shear stresses according to a new inverse hyperbolic shear deformation theory is studied in [14]. The proposed theory based upon the shear strain shape function yields non-linear distribution of transverse shear stresses and also satisfies traction-free boundary conditions. A new trigonometric shear deformation theory for the finite element analysis of thick multilayered plates is presented in [15]. The discrete element chosen is a four-node quadrilateral with seven degrees of freedom at each node. A new trigonometric higher order shear deformation theory, which accounts for an adequate distribution of the transverse shear strain throughout the plate thickness, and tangential stress-free boundary conditions on the plate boundary surface, is developed in [16] for isotropic and composite laminated and sandwich plates. A new higher order shear deformation theory, which accounts for an adequate distribution of the transverse shear strain throughout the plate thickness, and tangential stress-free boundary conditions on the plate boundary surface, is presented in [17] for elastic composite/sandwich plates and shells. The principle of virtual work was employed to derive the governing equations and boundary conditions. The equations were solved via a Navier-type, closed form solution. The analysis of thick orthotropic laminated composite plates based on higher order shear deformation theory, which accounts for parabolic distributions of transverse shear stresses and requires no shear correction factors, is reported on in [18]. A finite element program was developed using a serendipity element with seven degrees of freedom at each node. The bending, vibration and buckling behaviour of simply supported thick orthotropic rectangular plates and laminates is studied in [19] based on a three-dimensional, linear, small deflection theory of elasticity solution. The solution leads to simple infinite series for stresses and displacements in flexure, forced vibration, and to the closed-form characteristic equations for free vibration and buckling problems. Static behaviour of moderately thick anti-symmetrical angle-ply laminated shear-flexible clamped plates subjected to uniform distributed loads is examined in [20] using analytical and finite element techniques. A four-node quadrilateral element was used for the finite element solution, and Navier approach based on Reissner-Mindlin shear deformation theory was used for analytic investigation.

A new functional for the static analysis of symmetric cross-ply laminated thick plates is constructed in this research using a systematic procedure based on the Gâteaux differential. To the best of the authors' knowledge, the functional of laminated composite thick plates based on the mixed finite element method has not as yet been reported. The application of an efficient and simple method for the static analysis of symmetric cross-ply laminated composite thick plates will be presented in this paper based on the Gâteaux differential approach. The Gâteaux differential method is a more powerful, reliable and efficient variational tool compared to the widely accepted Hellinger-Reissner (HR) and Hu-Washizu (HW) variational principles, and it has many advantages over these principles. The mentioned techniques are extensively compared in [21] and [22]. The Gâteaux differential method was first used in [23]. Recently, Aköz and his co-workers have successfully employed the Gâteaux differential method to construct the functional that can be applied to a variety of problems [24-29]. The mathematical procedure and advantages of this method are explained in detail in a well-documented study presented in [27]. For the sake of simplicity, some significant advantages of the Gâteaux differential approach can briefly be summarized as follows:

- All field equations can be enforced to the functional systematically. For instance, the conventional variational principles HR and HW cannot be applied to any field equations for which stationary functional and boundary condition terms of the problem under consideration are not known beforehand, whereas the Gâteaux differential method can be applied. The mathematical procedure is explained in detail in [30].
- Geometric and dynamic boundary conditions can easily be obtained. For the HR and HW principles, the boundary conditions can be included in the functional by the Lagrange multipliers, whereas in the Gâteaux differential method boundary conditions are completely included by mathematical manipulations. For a detailed discussion about various variational principles, the reader is referred to papers [22, 31].
- Field equations can be checked using the potential test described in [34]. The Gâteaux differential method can be applied if the operator form of the field equations is potential (positive definite and self-adjoint). In the procedures involving conventional variational principles (HR and HW), an experience is required to select the parameters and sign of the boundary condition terms.
- The shear locking phenomenon is completely eliminated via the developed mixed element. Previous studies presented by Aköz and his co-workers have shown that shear locking is avoided if this approach is used [30, 32].

A new mixed-type finite element composed of four nodes, each with eight degrees of freedom, is formulated for this analysis. The proposed mixed finite element formulation is implemented using the FORTRAN software. This comprehensive parametric study is performed for evaluating the influences of the lamination scheme, number of layers, side-to-thickness ratio, boundary conditions, and geometrical and material properties variations on the displacement and stress components. The validity of the presented mixed finite element formulation is demonstrated by comparing the results with those available in the literature. Numerical comparison shows that the results obtained in this study agree well with those available in the literature.

2. Field equations

Consider a laminated plate as shown in Figure 1, having a rectangular cross-section of thickness h and consisting of k orthotropic layers.

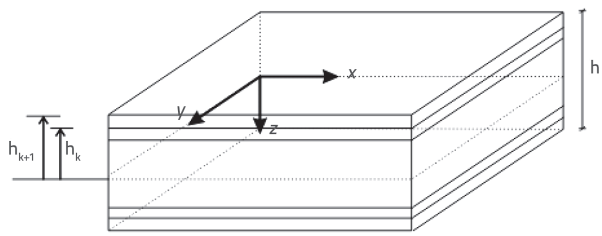

Figure 1. k-layers cross-ply laminated composite plate

A Cartesian coordinate system (x, y, and z) is defined at the central axis of the plate. Field equations of thick plates can be found in [33] and, referring to this study, field equations of symmetric cross-ply laminated composite plates can be written as:

$$
\begin{aligned}
&\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x=0\\
&\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-Q_y=0\\
&\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+q=0\\
&\frac{\partial \Omega_x}{\partial x}-\frac{Q_{11}}{E_x D_{11}}\left(M_x-\mu_{xy}M_y-\frac{qh^2}{10}\mu_{xz}q\right)=0\\
&\frac{\partial \Omega_y}{\partial y}-\frac{Q_{22}}{E_y D_{22}}\left(M_y-\mu_{yx}M_x-\frac{qh^2}{10}\mu_{yz}q\right)=0\\
&\frac{\partial \Omega_x}{\partial y}+\frac{\partial \Omega_y}{\partial x}-\frac{Q_{11}}{G_{xy}D_{11}}M_{xy}=0\\
&\Omega_x+\frac{\partial w}{\partial x}-\frac{6}{5G_{xz}h}Q_x=0\\
&\Omega_y+\frac{\partial w}{\partial y}-\frac{6}{5G_{yz}h}Q_y=0
\end{aligned}
\qquad (1)
$$

where, q is the uniform load applied on the top surface of the laminated plate. M_x, M_y, M_{xy}, Q_x and Q_y are the moment resultants and transverse force resultants, whose positive directions are shown in Figure 2. w and Ω_x are geometric parameters denoting the displacement of a point (z) in the laminated plate and the cross-sectional rotation about x axis.

The flexural rigidity matrix, D_{ij}, in the field equations is defined as:

$$D_{ij} = \frac{1}{3}\sum_{k-1}^{N}(\overline{Q}_{ij}^{(k)})\cdot(z_{k+1}^3 - z_k^3) \quad (i, j = 1, 2, 6) \tag{2}$$

where z_{k+1} and z_k are the coordinates of the upper and lower surfaces of the k[th] layer.

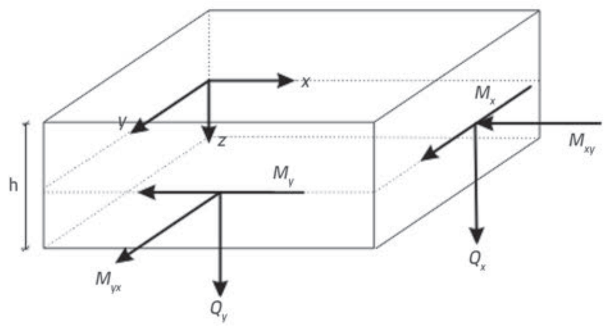

Figure 2. Internal forces

For orthotropic materials of each layer of laminated composite plates, the two-dimensional stress-strain equations for the k[th] layer can be written, under assumption of plane stress, as follows:

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix}^{(k)} = \begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}^{(k)} \begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix}^{(k)} \tag{3}$$

where σ_1, σ_2, τ_{12} are the stresses and ε_1, ε_2, γ_{12} are the linear strain components referred to the principal material coordinates of a layer. $Q_{ij}^{(k)}$ are elastic constants of the k[th] layer. They can be written as follows:

$$\begin{aligned} Q_{11} &= \frac{E_1}{(1-\mu_{12}\mu_{21})}\\ Q_{12} = Q_{21} &= \frac{\mu_{12}E_2}{(1-\mu_{12}\mu_{21})}\\ Q_{22} &= \frac{E_2}{(1-\mu_{12}\mu_{21})}\\ Q_{66} &= G_{12} \end{aligned} \tag{4}$$

with E_i being the Young's modulus in the i[th] material direction, μ_{ij} is the Poisson ratio, and G_{ij} is the shear modulus of the i-j plane. Constitutive equations for the k[th] orthotropic layer are transformed to the laminate coordinates (x, y, z) as follows:

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}^{(k)} = \left[\overline{Q}_{ij}\right]^{(k)}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}^{(k)} = \begin{bmatrix}\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16}\\ \overline{Q}_{21} & \overline{Q}_{22} & \overline{Q}_{26}\\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66}\end{bmatrix}^{(k)} \begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}^{(k)} \tag{5}$$

where $\overline{Q}_{ij}^{(k)}$ are the transformed elastic constants or stiffness matrix with respect to the laminate coordinates (x, y, z) [2]. The elements of the $\overline{Q}_{ij}$ matrix can be written as follows:

$$\begin{aligned} \overline{Q}_{11} &= Q_{11}\cos^4\theta + Q_{22}\sin^4\theta + 2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta\\ \overline{Q}_{12} &= (Q_{11}+Q_{22}-4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\sin^4\theta+\cos^4\theta)\\ \overline{Q}_{16} &= (Q_{11}-Q_{12}-2Q_{66})\sin\theta\cos^3\theta + (Q_{12}-Q_{22}+2Q_{66})\sin^3\theta\cos\theta\\ \overline{Q}_{22} &= Q_{11}\sin^4\theta + Q_{22}\cos^4\theta + 2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta\\ \overline{Q}_{26} &= (Q_{11}-Q_{12}-2Q_{66})\sin^3\theta\cos\theta + (Q_{12}-Q_{22}+2Q_{66})\sin\theta\cos^3\theta\\ \overline{Q}_{66} &= (Q_{11}+Q_{22}-2Q_{12}-2Q_{66})\sin^2\theta\cos^2\theta + Q_{66}(\sin^4\theta+\cos^4\theta) \end{aligned} \tag{6}$$

where θ is the angle between the global axis and the local x-axis of each layer.

3. The Gâteaux differential and functional

Field equations of composite laminated thick plates can be written in operator form as:

$$\boldsymbol{Q} = \boldsymbol{L}\boldsymbol{y} - \boldsymbol{f} \tag{7}$$

where $\boldsymbol{L}$ represents the coefficient matrix, $\boldsymbol{y}$ represents unknown vectors ($\boldsymbol{y} = \{w, M_x, M_y, M_{xy}, Q_x, Q_y, \Omega_x, \Omega_y\}$),, and $\boldsymbol{f}$ represents the load vector. The dynamic and geometric boundary conditions of composite plates can be written in symbolic form as follows:

Dynamic boundary conditions

$$\begin{aligned} M - \hat{M} &= 0\\ Q - \hat{Q} &= 0 \end{aligned} \tag{8}$$

Geometric boundary conditions

$$\begin{aligned} -\Omega + \hat{\Omega} &= 0\\ -w + \hat{w} &= 0 \end{aligned} \tag{9}$$

The Gâteaux derivative of an operator is defined as:

$$dQ(u;\overline{u}) = \frac{\partial Q(u+\tau\overline{u})}{\partial\tau}\Big|_{\tau=0} \tag{10}$$

where τ is a scalar. A necessary and sufficient condition that the operator $\boldsymbol{Q}$ be a potential [34] is:

$$\langle dQ(y;\overline{y}), y^*\rangle = \langle dQ(y;y^*), \overline{y}\rangle \tag{11}$$

where the inner products are shown in parentheses. After satisfying the requirement, the functional corresponding to the field equations is obtained as:

$$\mathbf{I}(\mathbf{y}) = \int_0^1 \langle \mathbf{Q}(s\mathbf{y},\mathbf{y}),\mathbf{y} \rangle \, ds \tag{12}$$

where s is a scalar quantity. The explicit form of the functional corresponding to the field equations of the cross-ply laminated composite thick plates is obtained as:

$$\begin{aligned}
\mathrm{I}(\mathbf{y}) = {} & [M_x,\Omega_{x,x}] + [M_{xy},\Omega_{x,y}] + [M_{xy},\Omega_{y,x}] + [M_y,\Omega_{y,y}] \\
& + [Q_x,\Omega_x] + [Q_y,\Omega_y] + [Q_x,w_{,x}] + [Q_y,w_{,y}] - [q,w] \\
& - \frac{Q_{11}}{2E_xD_{11}}\left\{[M_x,M_x] - 2\mu_{xy}[M_x,M_y] + [M_y,M_y] + 2(1+\mu_{xy})[M_{xy},M_{xy}]\right\} \\
& + \frac{Q_{11}h^2}{E_xD_{11}10}\left\{\mu_{xz}[q,M_x] + \mu_{yz}[q,M_y]\right\} - \frac{3}{5G_{xz}h}[Q_x,Q_x] - \frac{3}{5G_{yz}h}[Q_y,Q_y] \\
& - [\hat{M},\Omega]_\sigma - [\hat{Q},w]_\sigma - [(w-\hat{w}),Q]_\varepsilon - [(\Omega-\hat{\Omega}),M]_\varepsilon
\end{aligned} \tag{13}$$

The quantities with hat are known values at the boundaries. The parentheses with subscripts ε and σ indicate geometric and dynamic boundary conditions, respectively. Explicit expressions of boundary conditions are:

$$\begin{aligned}
[Q,w] &= [(Q_x n_x + Q_y n_y),w] \\
[M,\Omega] &= [(M_x n_x + M_{xy} n_y),\Omega_x] + [(M_{xy} n_x + M_y n_y),\Omega_y]
\end{aligned} \tag{14}$$

4. Finite element formulation

If the rectangular master element from Figure 3, with the parent shape function:

$$\Psi_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i) \qquad \xi_i,\ \eta_i = \pm 1;\ i = 1, \ldots, 4 \tag{15}$$

is used, all unknown variables of the functional are expressed in terms of interpolation functions, as shown below

$$w = \sum_{i=1}^{n} w_i \Psi_i \tag{16}$$

where n is the number of finite element nodes. Four-node serendipity element with eight degrees of freedom at each node is used.

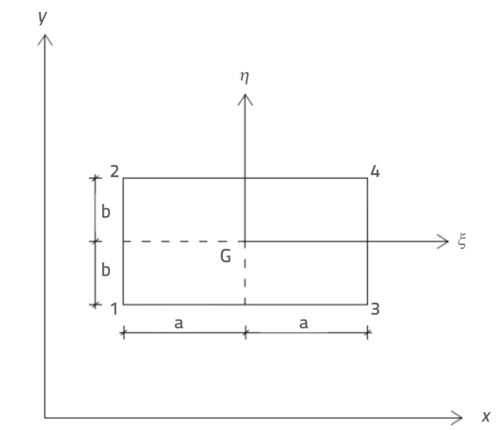

Figure 3. Rectangular master element

After substituting the above definitions into Eq. (13), the element matrix and load vector is derived explicitly:

$$[K] = \begin{array}{c} \begin{matrix} M_x & M_y & M_{xy} & Q_x & Q_y & \Omega_x & \Omega_y & w \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{matrix} \\ \begin{bmatrix}
2\gamma_1[k_1] & \gamma_2[k_1] & 0 & 0 & 0 & [k_2]^T & 0 & 0 \\
\gamma_2[k_1] & 2\gamma_3[k_1] & 0 & 0 & 0 & 0 & [k_3]^T & 0 \\
0 & 0 & 2\gamma_4[k_1] & 0 & 0 & [k_3]^T & [k_2]^T & 0 \\
0 & 0 & 0 & 2\gamma_5[k_1] & 0 & [k_1] & 0 & [k_2]^T \\
0 & 0 & 0 & 0 & 2\gamma_6[k_1] & 0 & [k_1] & [k_3]^T \\
[k_2] & 0 & [k_3] & [k_1] & 0 & 0 & 0 & 0 \\
0 & [k_3] & [k_2] & 0 & [k_1] & 0 & 0 & 0 \\
0 & 0 & 0 & [k_2] & [k_3] & 0 & 0 & 0
\end{bmatrix} \end{array} \tag{17}$$

$$\begin{bmatrix} \gamma_7[k_1]q \\ \gamma_8[k_1]q \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ [k_1]q \end{bmatrix} \tag{18}$$

where

$$\begin{aligned}
\gamma_1 &= -\frac{Q_{11}}{2E_xD_{11}}, & \gamma_5 &= -\frac{3}{5G_{xz}h} \\
\gamma_2 &= \frac{Q_{11}}{E_xD_{11}}\mu_{xy}, & \gamma_6 &= -\frac{3}{5G_{yz}h} \\
\gamma_3 &= -\frac{Q_{11}}{2E_xD_{11}}\frac{\mu_{xy}}{\mu_{yx}}, & \gamma_7 &= -\frac{1}{E_x}\frac{Q_{11}}{D_{11}}\mu_{xz}\frac{h^2}{10} \\
\gamma_4 &= -\frac{Q_{11}}{2D_{11}G_{xy}}, & \gamma_8 &= -\frac{1}{E_y}\frac{Q_{22}}{D_{22}}\mu_{yz}\frac{h^2}{10}
\end{aligned} \tag{19}$$

The explicit form of the quadrilateral element submatrices $[k_1]$, $[k_2]$, $[k_3]$, and the operator form of field equations, are given in Appendix I.

5. Numerical examples

Performance of the method is tested by means of the following plate problems. In all applications, a quarter of a rectangular plate is used, as shown in Figure 4. In addition, all layers are assumed to have the same thickness and orthotropic material properties.

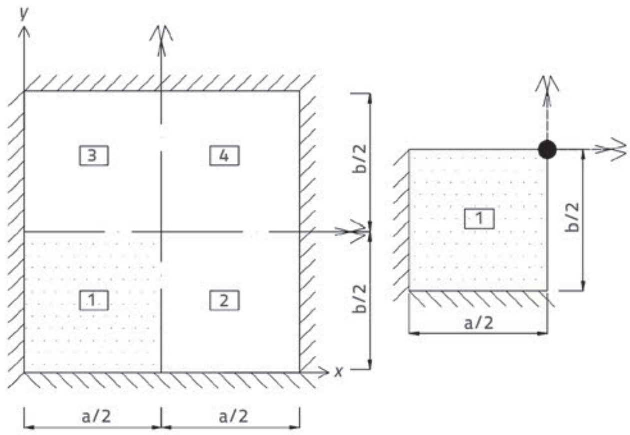

Figure 4 Geometrical properties of simply supported rectangular plate

5.1. Example 1

An orthotropic simply supported thick square plate subjected to uniformly distributed step load is considered. To select a suitably refined mesh scheme, the static results of the (0^0) laminated plates are considered using different orders of the mesh scheme. Based on different orders of the mesh scheme, the number of elements increased as shown in the first column of Table 1.

Material properties are:

$E_x / E_y = 25$; $\mu_{xy} = \mu_{xz} = \mu_{yz} = 0{,}25$; $G_{xy} = G_{xz} = 0{,}5\ E_y$; $G_{yz} = 0{,}2\ E_y$

Central displacement (w) and bending moments (M_x, M_y) of a one-layer laminated plate are computed for the span to thickness ratio (thickness parameter), a/h = 5. The results are presented in dimensionless form and compared to the existing results shown in the literature. The dimensionless equations of vertical displacement and bending moments are as follows:

$$w\left(\frac{a}{2},\frac{a}{2}\right) \rightarrow \bar{w} = w\frac{E_y h^3}{q_0 a^4}10^2$$
$$M\left(\frac{a}{2},\frac{a}{2}\right) \rightarrow \bar{M} = \frac{M}{q_0 a^2} \quad (20)$$

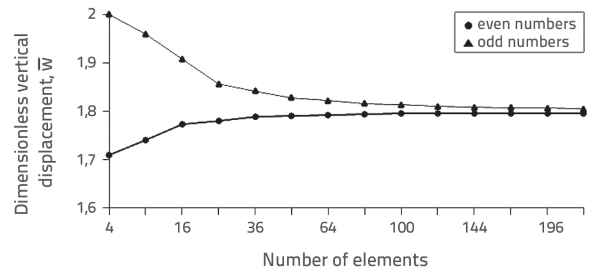

Figure 5. Convergence test of simply supported one-layer (0°) laminated plate for displacement

The dimensionless vertical displacement versus the number of elements is shown in Figure 5. The numerical results converge from above and below, depending on the odd and even number of elements. The convergence of shear forces is very similar to the convergence of bending moment and vertical displacement, and so the results are illustrated for vertical displacement only. It can be seen from Table 1 that the present model yields results comparable to those given in [35, 36].

Table 1. Dimensionless vertical displacement ($\bar{w}$) and bending moment values ($\bar{M}_x$, $\bar{M}_y$) of simply supported thick plate for different number of elements

Number of elements	$\bar{w}$			$\bar{M}_x$			$\bar{M}_y$		
	Present	[35]	[36]	Present	[35]	[36]	Present	[35]	[36]
4	1.7090	1.7097	1.8159	0.1138	0.1138	0.1206	0.0085	0.0085	0.0109
9	1.9590	1.9584		0.1315	0.1316		0.0125	0.0125	
16	1.7740	1.7746		0.1189	0.1189		0.0100	0.0100	
25	1.8560	1.8565		0.1245	0.1246		0.0115	0.0115	
36	1.7880	1.7885		0.1199	0.1199		0.0104	0.0104	
49	1.8280	1.8271		0.1226	0.1226		0.0112	0.0112	
64	1.7920	1.7931		0.1202	0.1202		0.0106	0.0106	
81	1.8160	1.8163		0.1218	0.1218		0.0111	0.0111	
100	1.7950	1.7947		0.1204	0.1203		0.0107	0.0107	
121	1.8100	1.8101		0.1214	0.1214		0.0110	0.0110	
144	1.7960	1.7962		0.1204	0.1204		0.0107	0.0107	
169	1.8070	1.807		0.1212	0.1212		0.0110	0.0110	
196	1.7960	1.7963		0.1205	0.1205		0.0108	0.0108	
225	1.8050	1.8039		0.1211	0.1206		0.0109	0.0109	

5.2. Example 2

An orthotropic simply supported cross-ply (0°/90°/0°) laminated square plate subjected to uniformly distributed load is considered. A uniform mesh with 225 elements (15x15) is used in a quarter of the plate.

Material properties are:

$E_1 = 25E_2$; $G_{12} = G_{13} = 0{,}5E_2$; $\mu_{12} = 0{,}25$

The dimensionless vertical displacement ($\bar{w}$) of the (0°/90°/0°) laminated thick plate, with various values of thickness parameter a/h, are compared with previously published results [2, 37], as shown in Table 2. As can be seen in the table, the results obtained using the present mixed finite element method closely agree with available solutions. In general, the bending moment and shear force results are also characterized by good accuracy, but the results are presented for vertical displacement only. The dimensionless equation of vertical displacement can be presented as follows:

$$w\left(\frac{a}{2},\frac{a}{2}\right) \rightarrow \quad \bar{w} = w\frac{E_2h^3}{q_0a^4}10^2 \qquad (21)$$

Table 2. Dimensionless vertical displacement ($\bar{w}$) values of simply supported thick plate for different values of thickness parameter, a/h

$\frac{a}{h}$	$\bar{w}$		
	Present	[2]	[37]
10	1.0160	1.0219	1.0221
20	0.7548	0.7572	0.7574
100	0.6681	0.6697	0.6698

5.3. Example 3

A clamped symmetric cross-ply (0°/90°/0°) laminated orthotropic thick plate subjected to uniform load (p_0) is considered. The dimensionless vertical displacement ($\bar{w}$) value of the laminated thick plate is calculated using a uniform mesh with 169 elements (13x13) for a quarter plate. The dimensionless equation for this example is as follows:

$$w\left(\frac{a}{2},\frac{a}{2}\right) \rightarrow \bar{w} = w\left(\frac{10^6}{p_0}\right) \qquad (22)$$

The material properties are:

$E_1 = 20{,}83 \times 10^6$ psi(per square inch), $E_2 = 10{,}94 \times 10^6$ psi, $G_{12} = 6{,}1 \times 10^6$ psi, $G_{13} = 3{,}71 \times 10^6$ psi, $G_{23} = 6{,}19 \times 10^6$ psi, $\mu_{12} = \mu_{13} = \mu_{23} = 0{,}44$, where is 1 *psi* = $6{,}895 \cdot 10^3$ [N/m²]

Numerical results of the dimensionless central displacement ($\bar{w}$) are presented in Table 3 for different thickness-to-span ratios (h/a) and length-to-width ratios (aspect ratio: a/b), and are compared with the corresponding results available in the literature [38]. Although the method used in the paper presents good results for the bending moment and shear force values, vertical displacement results are presented in this example for comparison. As can be seen in the table, the results of the presented procedure agree well with those available in the literature, and so the methodology of this study is considered to be reliable. In addition, it is clearly shown that the increasing value of aspect ratio of the plate a/b always decreases the deflection.

Table 3. Dimensionless vertical displacement ($\bar{w}$) values of clamped thick plate with different h/a and a/b ratios

$\frac{a}{b}$	$\frac{h}{a}$	$\bar{w}$	
		Present	[38]
1	0.2	42.1000	41.7590
	0.14	93.8200	92.7280
3	0.2	3.8680	3.8696
	0.14	6.7160	6.6931
5	0.2	1.2160	1.2192
	0.14	1.8900	1.8902

5.4. Example 4

The solution of the symmetric cross-ply (0°/90°/90°/0°) laminated plate with two opposite edges simply supported and other two edges clamped under uniformly distributed load is considered. A uniform mesh with 169 elements (13x13) is used in a quarter of the plate.

The corresponding material properties are:

E_x/E_y=1; 10 (individually considered); $G_{xy} = G_{xz} = 0{,}5E_y$; $G_{yz} = 0{,}2E_y$; $\mu_{xy} = \mu_{xz} = \mu_{yz} = 0{,}3$

The results of the dimensionless central displacement, bending moment, and shear force values are computed using the dimensionless equations as given in Eq. (23) for different span to thickness ratios of the plate a/h, as presented in Table 4.

$$\begin{aligned} &w\left(\frac{a}{2},\frac{a}{2}\right) \rightarrow \quad \bar{w} = w\frac{E_yh^3}{q_0a^4}10^2 \\ &M\left(\frac{a}{2},\frac{a}{2}\right) \rightarrow \quad \bar{M} = \frac{M}{q_0a^2} \\ &Q\left(\frac{a}{2},0\right) \rightarrow \quad \bar{Q} = \frac{Q}{q_0a} \end{aligned} \qquad (23)$$

It can be seen from the results that the increasing value of the E_x/E_y ratio always increases the values of M_x and Q_x and decreases the values of M_y and Q_y.

Table 4. Dimensionless vertical displacement, bending moment and shear force values of two opposite edges simply supported and other two edges clamped, under uniformly distributed load for different a/h ratios

a/h	$E_x / E_y = 1$					$E_x / E_y = 10$				
	$\bar{w}$	$\bar{M}_x$	$\bar{M}_y$	$\bar{Q}_x$	$\bar{Q}_y$	$\bar{w}$	$\bar{M}_x$	$\bar{M}_y$	$\bar{Q}_x$	$\bar{Q}_y$
5	3.2510	0.0292	0.0326	0.2682	0.4632	1.8510	0.0667	0.0271	0.3422	0.3676
10	2.3320	0.0248	0.0318	0.2589	0.4964	1.1000	0.0656	0.0262	0.3441	0.4001
20	2.0810	0.0235	0.0316	0.2558	0.5085	0.9074	0.0651	0.0261	0.3450	0.4199
50	2.0080	0.0231	0.0315	0.2554	0.5132	0.8524	0.0648	0.0261	0.3454	0.4272
100	1.9980	0.0230	0.0315	0.2554	0.5141	0.8445	0.0648	0.0261	0.3456	0.4284

6. Conclusion

The static analysis of symmetric cross-ply laminated composite thick plates is carried out using a mixed finite element method. Based on the Gâteaux differential approach, a new functional is developed for the laminated composite thick plates. Employing the developed mixed finite element formulation, symmetric cross-ply laminated composite thick plates, with a varying number of layers, are taken as examples to numerically evaluate the variation of geometrical and material parameters, and boundary conditions with the moment and transverse force resultants and displacements. Comparison between results obtained in the present study and those obtained in literature reveals that they are in good agreement. Results in the numerical examples confirm the validity and efficiency of the present analysis. Based on theoretical considerations and numerical assessments of the developed mixed finite element, the following remarks can be made:

- A new functional has been constructed for cross-ply laminated composite thick plates through a systematic procedure based on the Gâteaux Differential. The functional has eight independent variables.
- In addition, geometric (essential) and dynamic (natural) boundary conditions have been obtained.
- A special mixed finite element program, with eight independent field variables, has been written. Since the first derivatives of the variables exist in the functional, the conforming element formulation for the shape function N must satisfy the $C^0(r)$ continuity only.
- The developed mixed finite element is capable of predicting displacements and internal forces directly without any mathematical operation. The mixed finite element is not affected by shear locking.

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Appendix I

Operator form of field equations $\boldsymbol{Q} = \boldsymbol{Ly}\text{-}\boldsymbol{f}$:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & P_{17} & P_{18} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_{24} & 0 & P_{26} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & P_{35} & P_{36} & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & P_{42} & 0 & P_{44} & P_{45} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & P_{53} & P_{54} & P_{55} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & P_{62} & P_{63} & 0 & 0 & P_{66} & 0 & 0 & 0 & 0 & 0 & 0 \\ P_{71} & 1 & 0 & 0 & 0 & 0 & P_{77} & 0 & 0 & 0 & 0 & 0 \\ P_{81} & 0 & 1 & 0 & 0 & 0 & 0 & P_{88} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ \Omega_x \\ \Omega_y \\ M_x \\ M_y \\ M_{xy} \\ Q_x \\ Q_y \\ w \\ \Omega \\ M \\ Q \end{bmatrix} = \begin{bmatrix} q \\ 0 \\ 0 \\ -A_1 \\ -A_2 \\ 0 \\ 0 \\ 0 \\ \hat{Q} \\ \hat{M} \\ -\Omega \\ -\hat{w} \end{bmatrix} \tag{A.1}$$

where

$$\begin{aligned} &P_{17} = -\frac{\partial}{\partial x}, \quad P_{18} = -\frac{\partial}{\partial y}, \quad P_{24} = -\frac{\partial}{\partial x}, \quad P_{26} = -\frac{\partial}{\partial y} \\ &P_{35} = -\frac{\partial}{\partial y}, \quad P_{36} = -\frac{\partial}{\partial x}, \quad P_{42} = \frac{\partial}{\partial x}, \quad P_{44} = -\frac{Q_{11}}{E_x D_{11}} \\ &P_{45} = \frac{Q_{11}}{E_x D_{11}}\mu_{xy}, \quad P_{53} = \frac{\partial}{\partial y}, \quad P_{54} = \frac{Q_{22}}{E_y D_{22}}\mu_{yx}, \quad P_{55} = -\frac{Q_{22}}{E_y D_{22}} \\ &P_{62} = \frac{\partial}{\partial y}, \quad P_{63} = \frac{\partial}{\partial x}, \quad P_{66} = -\frac{Q_{11}}{G_{xy} D_{11}}, \quad P_{71} = \frac{\partial}{\partial x} \\ &P_{77} = -\frac{6}{5G_{xz}h}, \quad P_{81} = \frac{\partial}{\partial y}, \quad P_{88} = -\frac{6}{5G_{yz}h} \\ &A_1 = \frac{Q_{11}}{E_x D_{11}}\frac{qh^2}{10}\mu_{xz}, \quad A_2 = \frac{Q_{22}}{E_y D_{22}}\frac{qh^2}{10}\mu_{yz} \end{aligned} \tag{A.2}$$

Submatrices for quadrilateral element are:

$$[k_1] = \int \hat{\Psi}_i \hat{\Psi}_j dA = \begin{bmatrix} 4ab/9 & 4ab/18 & 4ab/18 & 4ab/36 \\ 4ab/18 & 4ab/9 & 4ab/36 & 4ab/18 \\ 4ab/18 & 4ab/36 & 4ab/9 & 4ab/18 \\ 4ab/36 & 4ab/18 & 4ab/18 & 4ab/9 \end{bmatrix} \tag{A.3}$$

$$[k_2] = \int_A \frac{\partial \hat{\Psi}_j}{\partial x} \hat{\Psi}_i dA = \begin{bmatrix} -b/3 & -b/6 & -b/3 & -b/6 \\ -b/6 & -b/3 & -b/6 & -b/3 \\ b/3 & b/6 & b/3 & b/6 \\ b/6 & b/3 & b/6 & b/3 \end{bmatrix} \tag{A.4}$$

$$[k_3] = \int_A \frac{\partial \hat{\Psi}_i}{\partial y} \hat{\Psi}_j dA = \begin{bmatrix} -a/3 & -a/3 & -a/6 & -a/6 \\ a/3 & a/3 & a/6 & a/6 \\ -a/6 & -a/6 & -a/3 & -a/3 \\ a/6 & a/6 & a/3 & a/3 \end{bmatrix} \tag{A.5}$$